

EXISTENCE AND UNIQUENESS THEOREM FOR CONVEX POLYHEDRAL METRICS ON COMPACT SURFACES

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ABSTRACT. We state that any constant curvature Riemannian metric with conical singularities of constant sign curvature on a compact (orientable) surface S can be realized as a convex polyhedron in a (Riemannian or Lorentzian) space-form. Moreover such a polyhedron is unique, up to global isometries, among convex polyhedra invariant under isometries acting on a totally umbilical surface.

This general statement falls apart into 10 different cases. The cases when S is the sphere are classical.

1. ORIGINAL STATEMENT FOR THE SPHERE

Let P be (the boundary of) a convex polytope in the Euclidean space \mathbb{R}^3 . The induced metric on P is isometric to a flat metric on the sphere \mathbb{S}^2 , with conical singularities, corresponding to the vertices. As P is convex, the angle around each conical singularity is lower than 2π , and then the singular curvature (2π minus the cone-angle) is positive. For simplicity we will call such a metric on \mathbb{S}^2 a $(0, +)$ -metric. The following famous theorem of A.D. Alexandrov, proved in the early 1940s, states the converse property.

Theorem 1 (Alexandrov Theorem, [Ale05]). *Let m be a $(0, +)$ -metric on the sphere \mathbb{S}^2 . Then*

- (1) *There exists a convex polytope P in \mathbb{R}^3 such that P realizes m .*
- (2) *P is unique, up to congruences.*

(“Realizes” means that the induced metric on P is isometric to the metric g .) Of course, the uniqueness of P in the theorem is only among convex polytopes, as it is easy to construct non-convex polytopes with the same induced metric than a given convex one. The case of degenerated convex polytopes (doubly covered convex polygons) is considered in the theorem above.

A convex polytope can be defined in any simply connected space of constant curvature. Let us restrict our attention to curvatures 1 and -1 . The original proof of the preceding theorem extends easily to these cases. The induced metric on a convex polytope in the sphere \mathbb{S}^3 (resp. the hyperbolic space \mathbb{H}^3) is isometric to a $(1, +)$ -metric (resp. a $(-1, +)$ -metric) on the sphere \mathbb{S}^2 : a spherical (resp. hyperbolic) metric with conical singularities of positive curvature. In the case of spherical metrics we also suppose that the number of conical singularities is > 2 , otherwise it is easy to construct examples for which the uniqueness part of the statement below is false [Ale05].

Theorem 2 ([Ale05]). **S:** *Let m be a $(1, +)$ -metric on the sphere. Then*

- (1) *There exists a convex polytope P in \mathbb{S}^3 such that P realizes m .*
- (2) *P is unique, up to congruences.*

H: *Let m be a $(-1, +)$ -metric on the sphere. Then*

- (1) *There exists a convex polytope P in \mathbb{H}^3 such that P realizes m .*
- (2) *P is unique, up to congruences.*

It is easy to visualize convex polytopes in \mathbb{S}^3 or \mathbb{H}^3 . Actually those spaces can be defined as (pseudo)-spheres in a vector space:

$$\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}, \mathbb{H}^3 = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1, x_1 > 0\}.$$

A projection from the origin of (a part) of these spaces onto any affine plane gives an affine representation of these spaces seen as subspaces of the projective space of dimension 3. And a convex polytope is a projective object. Of course, the induced metric is not a projective invariant, but from this observation some generalizations of the theorems above become natural.

Date: November 16, 2010.

For the Proceedings of the Conference Efimov-100. The author want to thank the organizers, in particular Sergey Mikhalev and Idjad Sabitov. The exposition was improved thanks to comments of Alexander A. Borisenko, Ivan Izmetiev and Jean-Marc Schlenker.

The classical way of proving these statements is to use a *continuity method*, for which existence part follows from the proof of uniqueness part using topological arguments [Ale05]. The uniqueness part is a refinement of the famous Cauchy Theorem about rigidity of convex polytopes. The proof of the latter is based on the so-called Cauchy Lemma, which is only of topological nature. The continuity method can be adapted such that an infinitesimal rigidity result suffices, instead of a rigidity result. And infinitesimal rigidity is also of projective nature. This has to be understood in the following sense. An infinitesimal isometric deformation of a convex polytope P (in any constant curvature pseudo-Riemannian manifold) can be defined as the restriction to each faces of P of Killing fields, such that they coincide on edges. If there exists a Killing field such that its restriction to P corresponds to the infinitesimal isometric deformation, then the deformation is said to be trivial, and if any deformation is trivial, P is infinitesimally rigid. But if there exists a map between two pseudo-Riemannian manifolds sending geodesics to geodesics (it is the case in particular for projective maps), then there exists a map sending Killing fields of one manifold to the Killing fields of the other. This result is based on properties of the connection [Kne30]. It was later rediscovered by Volkov [Vol74], and particular cases were known for a long time (Darboux–Sauer, Pogorelov, ...). There is another interpretation of the projective nature of infinitesimal rigidity of convex polytopes, based on similar results about frameworks, see [Izm09] and references therein.

A proof using the continuity method is not constructive, because it is based on the Domain Invariance Theorem. A constructive proof of the existence part of Theorem 1 was done in [BI08]. It does not require a proof of the uniqueness part. It uses a *variational method*, in which the desired polytopes is described as the critical point of a functional. The definition of this functional uses a polyhedral version of the total scalar curvature. Another proof was given in [Vol55], see the introduction of [BI08].

2. HYPERBOLIC EXTENSION

The most famous projective model of the hyperbolic space is the Klein model: the interior of the unit ball endowed with the Hilbert distance (Figure 1, left). Any (projective) convex polytope contained in the interior of the unit ball can be considered as a hyperbolic polytope. But one can look at the induced hyperbolic metric on the intersection of the unit ball with convex polytopes that are not totally contained in the unit ball (Figure 1, right). If each edge of the projective polytope meets the interior of the unit ball, we call its hyperbolic part a *generalized hyperbolic polyhedron*. The induced metric on the neighborhood of a vertex on the unit sphere is isometric to a hyperbolic cusp. If the vertex is out of the closed unit ball, the induced metric on the intersection of the faces meeting at this vertex with the hyperbolic space is isometric to a complete end of infinite area.

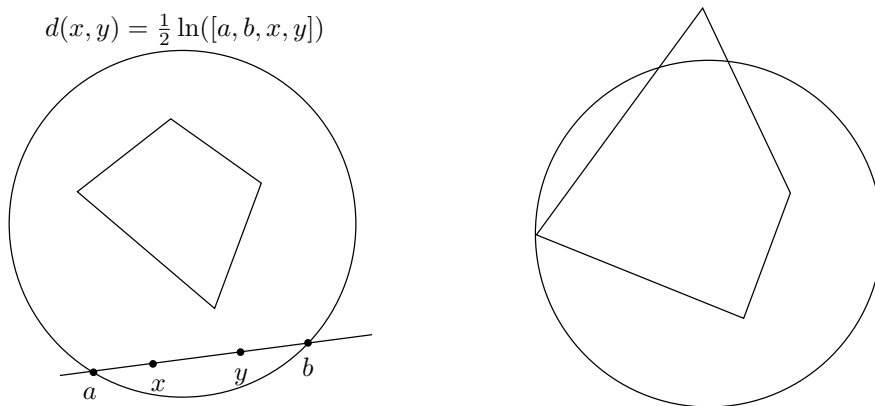


FIGURE 1. d is the Hilbert distance, with $[\cdot, \cdot, \cdot, \cdot]$ the cross-ratio.

Hyperbolic part of Theorem 2 can be extended in this way:

Theorem 3. *Let m be a hyperbolic metric on the (punctured) sphere \mathbb{S}^2 with conical singularities of positive singular curvature, cusps and complete ends of infinite area. Then*

- (1) *There exists a convex generalized hyperbolic polyhedron P in \mathbb{H}^3 such that P realizes m .*
- (2) *P is unique, up to congruences.*

The case with only cusps was proved in [Riv94]— this reference also contains the uniqueness part of the theorem above. The proof of the theorem is a part of a more general result contained in [Sch98a], see Section 9. A constructive proof for the case with cusps has been announced, see [BPS10].

3. DE SITTER EXTENSION

The next step is to realize Riemannian metrics in a Lorentzian space. The outside of the projective model of the hyperbolic space is a projective model of (a part of) de Sitter space, see Figure 2. De Sitter space is the simply-connected Lorentzian space with constant curvature 1. It can also be defined as a pseudo-sphere in the Minkowski space, endowed with the induced metric:

$$dS^3 = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1\}.$$

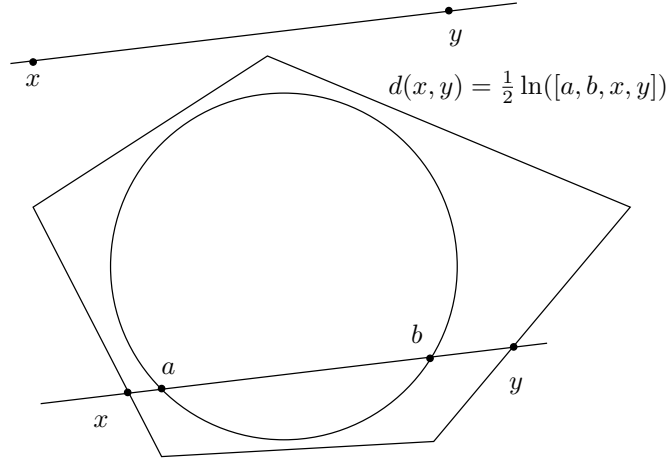


FIGURE 2. When the line don't meet the closed ball, the space is complexified. Then intersection points with the unit sphere exist, and a imaginary "distance" can be defined.

Consider a convex polytope P in \mathbb{H}^3 . In the projective Klein model, P has a polar dual with respect to the unit sphere, denoted by P^* . We can see P^* as a convex polyhedral surface in de Sitter space. The induced metric on P^* is isometric to a spherical (Riemannian) metric on the sphere \mathbb{S}^2 with conical singularities of negative curvature. Moreover the length of closed geodesics is $> 2\pi$. We call such a metric a *large* $(1, -)$ -metric on \mathbb{S}^2 . In particular the number of cone-points is > 2 . Actually the metric induced on the dual P^* of P is isometric to the dual metric, or Gauss image (the polyhedral analogue of the third fundamental form), of P . See [RH93, CD95].

Theorem 4 ([Riv86, RH93]). *Let m be a large $(1, -)$ -metric on the sphere. Then*

- (1) *There exists a convex polyhedral surface P in de Sitter space such that P realizes m .*
- (2) *P is unique, up to congruences.*

Actually the uniqueness statement proved in [RH93] is among convex polyhedral surface of de Sitter space which are polar dual of hyperbolic convex polytopes. The uniqueness among all convex polyhedral surfaces (homeomorphic to the sphere) of de Sitter space is a consequence of the results of [Sch01a] (in other words, only dual of convex polytopes of \mathbb{H}^3 have induced large $(1, -)$ -metric). This is not straightforward as there exists convex polytopes in de Sitter space (in the sense that they bound a compact set of de Sitter space) whose induced metric is Riemannian, see Figure 3. They are characterized by the following theorem.

Theorem 5 ([Sch01a]). *Let m be a metric on the sphere \mathbb{S}^2 such that:*

- *m is a spherical metric except at a finite number of singular points x_1, \dots, x_N ;*
- *$\mathbb{S}^2 = D_+ \cup D_-$, where D_+ and D_- are topological discs intersecting on a curve $C = \partial D_+ = \partial D_-$ of length $< 2\pi$ with vertices the singular points x_1, \dots, x_p ;*
- *the singular curvature of m at x_{p+1}, \dots, x_N is negative;*
- *D_+, D_- are convex for m , and strictly convex at each vertex of C ;*
- *geodesics segments of D_+, D_- have length less than π .*

Then

- (1) *There exists a convex polytope P in de Sitter space such that P realizes m .*
- (2) *P is unique, up to congruences.*

A similar statement holds in the Minkowski space $\mathbb{R}^{2,1}$.

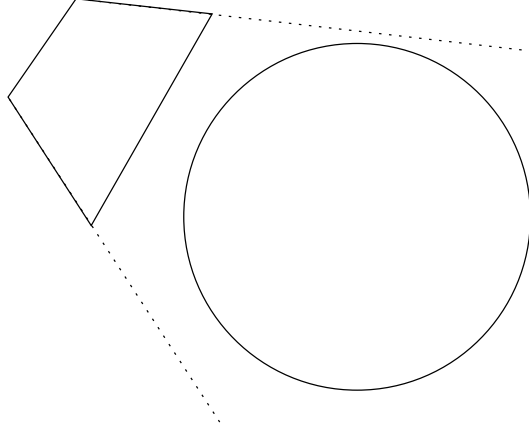


FIGURE 3. A convex polytope in de Sitter space whose induced metric is Riemannian, but which is not the dual of a hyperbolic convex polytope.

Theorem 6 ([IS90, Sch01a]). *Let m be a metric on the sphere \mathbb{S}^2 such that:*

- *m is a flat metric except at a finite number of singular points x_1, \dots, x_N ;*
- *$\mathbb{S}^2 = D_+ \cup D_-$, where D_+ and D_- are topological discs intersecting on a curve $C = \partial D_+ = \partial D_-$ with vertices the singular points x_1, \dots, x_p ;*
- *the singular curvature of m at x_{p+1}, \dots, x_N is negative;*
- *D_+, D_- are convex for m , and strictly convex at each vertex of C .*

Then

- (1) *There exists a convex polytope P in $\mathbb{R}^{2,1}$ such that P realizes m .*
- (2) *P is unique, up to congruences.*

As far I know, there does not exist a similar statement for convex polytopes with induced Riemannian metric in the *anti-de Sitter space*, which is a Lorentzian space-form (non-simply connected) of curvature -1 , which can be defined as the following pseudo-sphere, endowed with the induced metric:

$$\text{AdS}^3 = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1\}.$$

Note that the two last theorems are somewhat different of the preceding ones, as they don't describe the induced metric on all the convex polytopes (this one can be non Riemannian for a random convex polytope). Moreover the induced metrics have both positive and negative singular curvatures. Anyway they are specific to the case of the sphere.

4. HIGHER GENUS?

The previous theorems (1, 2, 4) are concerned with some (K, ϵ) -metric on the sphere, $K \in \{-1, 0, 1\}, \epsilon \in \{-, +\}$. For which K and ϵ such metrics exist? The answer is given by Gauss-Bonnet formula

$$(1) \quad \sum_i k_i = 2\pi(2 - 2g) - KA(S),$$

where g is the genus of the compact surface S ($g = 0$ in the present case S is the sphere), k_i are the singular curvatures and $A(S)$ is the area of S . Hence there is no other (K, ϵ) -metric on the sphere than the ones previously considered: $(0, +), (1, +), (-1, +), (1, -)$.

So a natural question is: what about realization of (K, ϵ) -metrics on compact surfaces of higher genus? The aim is to realize such metrics as convex polyhedra. Hence a (K, ϵ) -metric has to be realized in M_K^ϵ , where

$$M_0^+ = \mathbb{R}^3, M_1^+ = \mathbb{S}^3, M_{-1}^+ = \mathbb{H}^3, M_0^- = \mathbb{R}^{2,1}, M_1^- = \text{dS}^3, M_{-1}^- = \text{AdS}^3.$$

But each of these spaces (or parts of these spaces) has a model in an affine chart of the projective space, and in this space convexity and compactness implies genus 0. Then we will realize the universal cover of the metric. We will obtain convex polyhedral surface equivariant under the action of a representation of the fundamental group of the surface in the isometry group of (the suitable) M_K^ϵ .

We give some examples before definitions and statements.

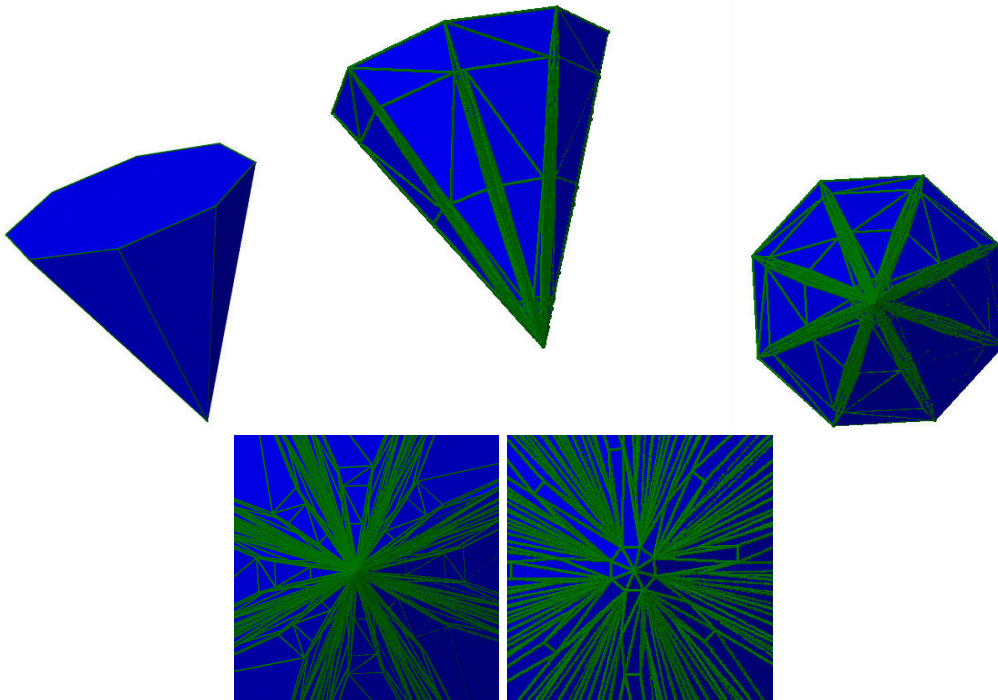


FIGURE 4. The first picture is the convex hull of $p = (0, 0, 1)$ together with its images for the generators of Γ (four elements together with their inverses). The second picture is the convex hull of p and points $\gamma_1 \circ \gamma_2 \circ \gamma_3(p)$, where γ_i is a generator of Γ (or the inverse of a generator). The height (the third coordinate) of p is 1. The height of $\gamma_1(p)$ is ≈ 10 . The height of $\gamma_1 \circ \gamma_2(p)$ is ≈ 100 . The height of $\gamma_1 \circ \gamma_2 \circ \gamma_3(p)$ is ≈ 1000 . The remaining pictures are zoomed views of the surface from the bottom. For precision reasons the vertices in the images above belongs to the hyperboloid of ray 1000. These objects can be manipulate at the author's web page. They were realized with Sage [S⁺10].

5. EXAMPLE OF FLAT SURFACES OF GENUS 2

The most simple example of compact hyperbolic surface is given by the quotient of \mathbb{H}^2 by a group of isometries whose a fundamental domain is a regular octagon (in the Klein projective model). Such a group is a Fuchsian group, that explains the terminology used below. This group has a representation Γ into the group of the isometries of Minkowski space preserving the upper-sheet of the hyperboloid.

Let $p \in \mathbb{H}^2$ seen as a vector of the Minkowski space. We call P the convex hull *in the ambient Minkowski space* of the orbit of p under the action of Γ (that gives an infinite number of points on the hyperboloid).

By construction, P is a convex polyhedral surface, invariant under the action of Γ , and it can be shown that the induced metric is Riemannian. The pair (P, Γ) is an example of *Fuchsian polyhedron*. The quotient P/Γ is isometric to a compact surface of genus 2 with a flat metric with one conical singularity of curvature $2\pi - 8 \times \frac{3\pi}{4} = -4\pi < 0$ (cone-angle 6π). Parts of such a polyhedral surface are drawn on Figure 4, with $p = (0, 0, 1)$. This is an example of the so-called *convex hull construction*, which appeared in the seminal papers [Pen87, EP88, NP91].

The definition of convex Fuchsian polyhedron in the Minkowski space is as follows. The vertices are orbits of points which can live on different upper-sheets of hyperboloids centered at the origin.

Definition 7. A convex Fuchsian polyhedron in the Minkowski space is a pair (P, Γ) , where

- P is a space-like (i.e. Riemannian) convex polyhedral surface in the Minkowski space $\mathbb{R}^{2,1}$,
- Γ is a group of isometries of $\mathbb{R}^{2,1}$ acting cocompactly on $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$ (i.e. \mathbb{H}^2/Γ is a compact hyperbolic surface),

such that Γ leaves P (globally) invariant.

And the corresponding existence and uniqueness statement reads as:

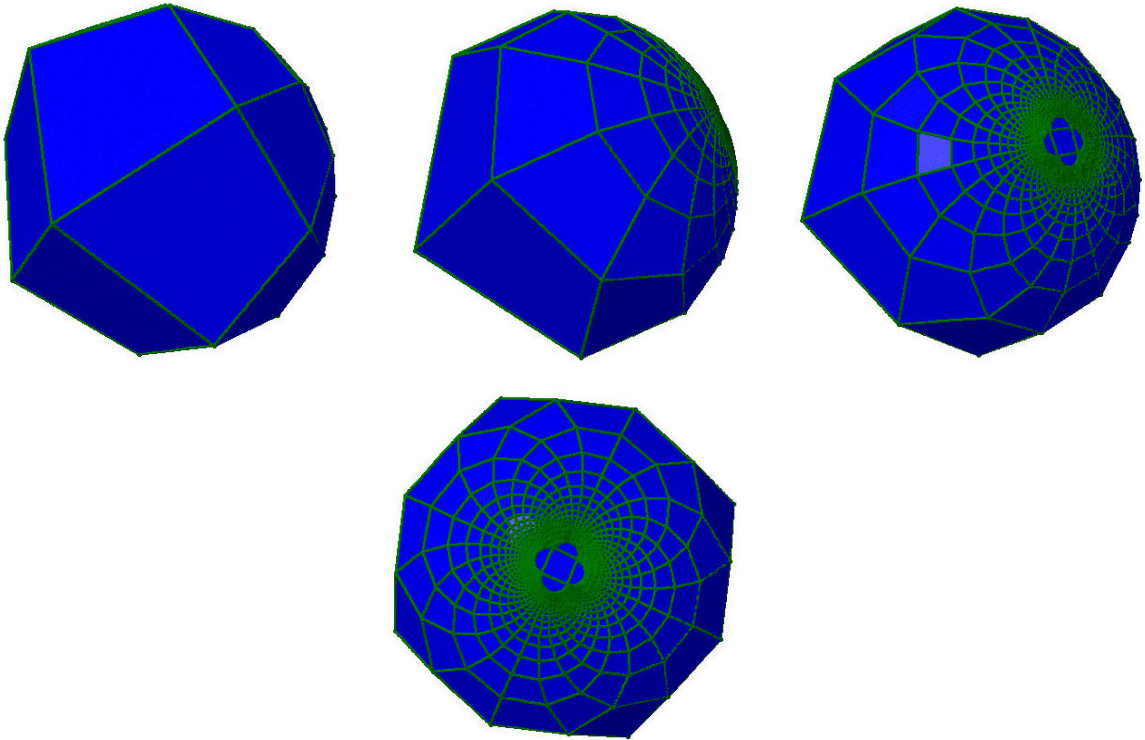


FIGURE 5. The parabolic polyhedron is made of an infinite number of vertices accumulating on a same point. (Only a finite number of vertices are represented on the picture.) This object can be manipulate at the author's web page. It was realized with Sage [S⁺10].

Theorem 8 ([Fil10]). *Let m be a $(0, -)$ -metric on a compact surface S of genus > 1 . Then*

- (1) *There exists a convex Fuchsian polyhedron (P, Γ) in $\mathbb{R}^{2,1}$ such that P realizes m (i.e. P/Γ endowed with the induced metric is isometric to (S, m)).*
- (2) *(P, Γ) is unique (among convex Fuchsian polyhedra), up to congruences.*

6. EXAMPLE OF HYPERBOLIC TORUS

The hyperbolic space has a model (the upper-space model) in which horizontal planes (in the upper part of space) are isometric to the Euclidean plane. They are examples of horospheres. Let us consider the plane H at height 1, and its tessellation by squares of unit length. The quotient of the Euclidean plane by the group whose fundamental domains are the squares is a flat torus. This group acting on the Euclidean plane extends to a group Γ of isometries of the hyperbolic space (the group is made of parabolic isometries). There is a map from this model of hyperbolic space to the Klein projective model, which sends H to an ellipsoid of rays $(1/2, 1/4)$, passing through the origin and tangent to the unit sphere. We also denote by H this ellipsoid. Let $p \in H$. The orbit Γp is made of an infinite number of points accumulating on the point of tangency between H and the unit sphere.

We call P the convex hull in the hyperbolic space (that is the same thing, in the Klein projective model, than the convex hull in \mathbb{R}^3) of Γp . By construction, P is a convex polyhedral surface, invariant under the action of Γ . All the faces are isometric hyperbolic squares. The pair (P, Γ) is an example of *parabolic polyhedron* of the hyperbolic space. The quotient P/Γ is isometric to a torus with a hyperbolic metric with one conical singularity of positive curvature (obtained by identifying opposed edges of any face). Such a polyhedral surface is drawn on Figure 5, with p the origin. Note that we can associate to the same polyhedral surface different parabolic polyhedra, by changing the group action, such that the different quotients are not isometric. The definition of convex parabolic polyhedron in the hyperbolic space is as follows. The vertices are orbits of points which can live on different ellipsoids having the same point of tangency with the unit sphere.

Definition 9. *A convex parabolic polyhedron in the hyperbolic space is a pair (P, Γ) , where*

- *P is a convex polyhedral surface in \mathbb{H}^3 ,*

- Γ is a group of isometries of \mathbb{H}^3 acting cocompactly on a horosphere, such that Γ leaves P (globally) invariant.

And the corresponding existence and uniqueness statement reads as:

Theorem 10 ([FI09]). *Let m be a $(-1, +)$ -metric on the torus T . Then*

- (1) *There exists a convex parabolic polyhedron (P, Γ) in \mathbb{H}^3 such that P realizes m (i.e. P/Γ endowed with the induced metric is isometric to (T, m)).*
- (2) *(P, Γ) is unique (among convex parabolic polyhedra), up to congruences.*

The uniqueness have to be among convex parabolic polyhedra, as there is another way to realize $(-1, +)$ -metrics on the torus. Let us call *banana* a surface of \mathbb{H}^3 at constant distance from a geodesic. The banana endowed with the induced metric is isometric to a complete infinite Euclidean cylinder. It is then easy to construct a polyhedral surface P and a group of isometries Γ acting cocompactly on the banana, such that P/Γ is isometric to a $(-1, +)$ -metric on the torus. If we call such a pair a *convex polyhedral banana*, the following question holds

Question 11. *Let m be a $(-1, +)$ -metric on the torus T . Does it exist a (unique) convex polyhedral banana realizing (T, m) ?*

It seems that there is no other way to obtain $(-1, +)$ -metrics on the torus, as horospheres and bananas are the only complete isometric immersions of the Euclidean plane into \mathbb{H}^3 [VV71, Sas73]. The same remarks can be done in the case of de Sitter space [DN81], in which horospheres and bananas can be defined similarly.

7. HOW MANY CASES?

What are the possible (K, ϵ) -metric on a surface of genus g ? A priori there are 18 cases: 3 choices for K , 2 choices for ϵ , and 3 choices for g (depending if the universal cover is a sphere, a disc or a plane). But by Gauss–Bonnet Formula (1), there are only 10 possible cases, which are listed in Table 1.

g	K	ϵ	DM	VM
0	-1	+	[Ale42]	[BI08]
0	0	+	[Ale05]	
0	1	+	[Ale05]	
0	1	-	[RH93]	
1	-1	+		[FI09]
1	1	-		[FI10]
≥ 2	-1	+	[Fil07]	
≥ 2	-1	-	[Fil10]	
≥ 2	0	-	[Fil10]	
≥ 2	1	-	[Sch04, Fil10]	

TABLE 1. The ten cases of realization of metrics of curvature K with cone singularities of curvature of sign ϵ on a compact surface of genus g . The column named by DM lists the articles using a deformation method and the column named by VM lists the articles using a variational method. The $(1, -)$ -metrics are supposed to be large.

8. THE GENERAL STATEMENT

The ten statements listed in Table 1 can be resumed in a unique statement. Let S be a compact surface of genus g , $K \in \{-1, 0, 1\}$ and $\epsilon \in \{-, +\}$. Sometimes $(0, +)$ -metrics on the sphere were called *convex polyhedral metrics* on the sphere, as by Alexandrov Theorem they correspond to induced metrics on convex polytopes in Euclidean space. By analogy we call any (K, ϵ) -metric a convex polyhedral metric, as the following statement says that they correspond to induced metrics on convex polyhedral surfaces.

Theorem 12 (Existence and uniqueness theorem for convex polyhedral metrics on compact surfaces). *Let m be a (K, ϵ) -metric on S (large for $(K, \epsilon = (1, -))$). Then*

- (1) *There exists a convex polyhedral surface $P \subset M_K^\epsilon$ and a group Γ of isometries $\mathbb{R}^{2,1}$, acting cocompactly on a totally umbilic surface, such that P/Γ is isometric to (S, m) .*
- (2) *(P, Γ) is the unique pair of this kind, up to congruences.*

If, for a pair (P, Γ) as in the statement above, the umbilic surface is isometric to the Euclidean plane, (P, Γ) is called a convex parabolic polyhedron, and if the umbilic surface is isometric to the hyperbolic plane (up to a homothety), (P, Γ) is called a convex Fuchsian polyhedron. If S is the sphere, the action of Γ is trivial.

In the $(-1, +)$ -cases, the statement above can be extended in the same way than Theorem 3.

Theorem 13 ([Fil08]). *Let m be a hyperbolic metric on a surface S of finite topological type with conical singularities of positive singular curvature, cusps and complete ends of infinite area. Then*

- (1) *There exists a convex generalized hyperbolic polyhedral surface P in \mathbb{H}^3 and a group Γ of isometries of \mathbb{H}^3 , acting cocompactly on a totally umbilic surface, such that P/Γ is isometric to (S, m) .*
- (2) *(P, Γ) is the unique couple of this kind, up to congruences.*

The uniqueness in the statement above is only among polyhedral surfaces invariant under the action of these groups acting on umbilic surfaces. In a certain sense they are the “most simple”. For example it is possible to construct convex polyhedral surfaces in \mathbb{H}^3 invariant under groups more complicated than Fuchsian groups (as for example quasi-Fuchsian groups), whose quotient provides $(-1, +)$ -metrics on compact surfaces of genus > 1 .

9. RELATED WORKS

Higher dimensions. The results presented here are specific to surfaces. A result such as Theorem 1 is possible because on one hand the space of convex polytopes with n vertices (up to congruences) has dimension $3n - 6$ (the polytope is determined by the positions of the vertices in space, and isometry group has dimension 6), and on the other hand the space of Euclidean metrics on the sphere with n conical singularities of positive curvature up to isometries has dimension $3n - 6$ (roughly speaking, the metric is determined by the conformal structure of the punctured sphere and the values of the singular curvatures [Tro86, Tro91]). Let us discuss for example the Euclidean case of dimension 4. Convex polytopes are still determined by their vertices, but the induced metric is singular along edges, so the space of polytopes and the space of metrics don’t have the same dimension. Moreover the uniqueness is obvious, at least locally (the link of a vertex is a convex polytope in \mathbb{S}^3 , which is rigid by Theorem 2, that implies that the Euclidean convex polytope is rigid).

Pseudo-Riemannian metrics on surfaces. In Theorem 13 we considered “generalized hyperbolic polyhedron”, which are the intersection of convex polyhedral surface of projective space with the unit ball, endowed with the Hilbert metric. But we could look at the induced metric on convex polyhedral surface in the “hyperboloid de Sitter” space (an extension of \mathbb{R}^3 endowed with the Hilbert metric). Such induced metric can be Riemannian, Lorentzian or degenerated on different faces and edges. For closed polyhedral surfaces with a finite number of vertices, this is done in [Sch98a, Sch01a]. The case of convex polytopes in Minkowski space is done in [Sch01a].

Circle patterns. The problems of circle patterns on surfaces have deep connections with the geometry of hyperbolic polyhedra. Let us consider the example of the flat torus. Roughly speaking, if a triangulation T of the torus together with numbers associated to the edges are given, satisfying some conditions, then there is a flat metric on the torus (unique up to scalar multiple), a geodesic triangulation isotopic to T and a (unique) family of circles, centered at the vertices of T , and intersecting at an angle given by the given numbers (this is a *circle pattern* on the torus). The relation between circle patterns and hyperbolic polyhedra is as follow. Consider a circle pattern of the torus is given, and look at the universal cover. Let us consider him as the horizontal plane at height zero in three-space, and draw above each circle a half-sphere. In the upper-space model of the hyperbolic space, those half-circles bound a convex parabolic polyhedron, and the angles between the circles correspond to dihedral angles of the polyhedron. The triangulation of the circle pattern is the dual of the cellulation of the polyhedron, hence it is simple, and then dihedral angles determine the dual metric: circle pattern problem is reduced to a realization problem. For example Theorem 4 can be considered as a generalization of the Koebe–Andreiev–Thurston theorem for circle pattern on the sphere (recall that realization of hyperbolic polyhedra with respect to the dual metric is the same as realization of de Sitter polyhedra with respect to the induced metric). There exists a variational proof for existence of circle pattern on compact surface [Thu97, CL03] but the functional was not explicitly known. Actually (in the torus case) this functional is the one appearing in the variational proof for realization theorem with respect to the dual metric [FI10]. It is related to the discrete Hilbert–Einstein functional and uses the volume of hyperbolic polyhedra.

The vertices of the hyperbolic polyhedra constructed from a circle pattern on a compact surface can be inside, on the boundary at infinity, or outside the hyperbolic space, depending on how the circles meet. Close to the scope of this text are the results of [BB02, Rou04], about circle patterns corresponding to hyperbolic polyhedra with vertices outside of hyperbolic space, but with edges meeting hyperbolic space (hyperideal polyhedra).

Circle patterns can be generalized to *inversive distance circle packings*, where the inversive distance is a quantity prescribed between two circles with maybe empty intersection. If the intersection is non-empty, the inversive distance is the intersection angle between the circles, see [Luo10] and references therein. If the circles are seen on the boundary at infinity of the hyperbolic space, the inversive distance between two non-intersecting circles is nothing but the hyperbolic distance between the two hyperbolic planes defined by the circles. It would be interesting to know if the results of [Luo10] could be expressed in terms of hyperbolic polyhedra (which are now allowed to have edges outside of hyperbolic space), and if the functionals used in this paper could be related to hyperbolic volume.

Connections between realization of ideal hyperbolic polyhedra and “discrete conformal maps” are done in [BPS10].

Convex metrics on surfaces. Let m be the $(0, +)$ -metric induced on a convex polytope of the Euclidean space. By the uniqueness part of Theorem 1, there does not exist other convex polytope (up to congruence) with induced metric m . But we can ask if there exists a convex surface with induced metric m (a convex surface in \mathbb{R}^3 is the boundary of a convex body). The answer is negative, as it was proved by Olovyanishnikov few times after Alexandrov proved Theorem 1. This result was extended by Pogorelov: a convex surface is uniquely determined by its induced metric [Pog73, Bus08].

The induced metric on a convex surface satisfies some convexity conditions. A.D. Alexandrov showed that any metric on the sphere satisfying such conditions is realized by a convex surface. The proof is obtained from Theorem 1. by approximation by polytopes. See [AZ67, Ale06] for more details. There does not exist such general results in other spaces as far I know.

Convex polyhedra can be seen as a particular case of convex surfaces. Another particular case are convex smooth surfaces, for which many results exist. The existence part of Theorem 1 is known as Weyl Problem and was solved by many authors, until papers of Nirenberg and Pogorelov in the 50’s. A variational proof was proposed in [BH37], and Ivan Izhestiev is currently working on it. The smooth analogue of Theorem 4 was done in [Sch94, Sch96]. [Sch98b] contains the smooth analogue of Theorem 3. The smooth analogue of Fuchsian results in the hyperbolic space is a consequence of the more general result of [Sch06] (see below). Fuchsian results in Lorentzian spaces were done in [LS00]. The idea of such results seems to go back to [Gro86, 3.2.4, B’, B’]. Smooth parabolic results are not known for the moment, but a positive answer to the smooth version of Question 11 is done in [Sch06]. Note that here “smooth” means C^∞ . For a review of results with assumptions about the regularity of the metric or of the immersion, see [IKS91, IKS95].

Hyperbolic manifolds with convex boundary. The hyperbolic part of Theorem 2 is equivalent to:

Theorem 14. *Let m be a $(-1, +)$ -metric on the sphere. Then there exists a unique hyperbolic metric on the ball B such that the induced metric on ∂B is isometric to m .*

Theorem 4 provides an analogous statement, with the induced Gauss metric instead of the induced metric on the boundary. Theorem 10 is equivalent to:

Theorem 15. *Let m be a $(-1, +)$ -metric on the torus. Then there exists a unique hyperbolic metric on a cusp C with convex polyhedral boundary such that the induced metric on ∂C is isometric to m .*

The de Sitter analogue of Theorem 10 provides a statement analogous to the one above [FI10]. A natural extension of such statements is to ask if, for a hyperbolic 3-manifold with convex boundary, the induced metric on the boundary uniquely determines the hyperbolic metric. Here there is no other assumption on the boundary than convexity. However a reasonable assumption on M should be that it admits a complete hyperbolic convex cocompact metric, i.e. M contains a non-empty, compact, geodesically convex subset. This can be thought as an extension of Mostow Rigidity Theorem (the topology and the induced metric on the boundary uniquely determines the hyperbolic metric on the manifold). A celebrated particular case is the one of induced metric on the boundary of the convex core of hyperbolic manifolds. In this case the metric on the boundary is hyperbolic. See the introduction of [Sch06] for references. Besides if the boundary is asked to be smooth and strictly convex, such a statement was done in [Sch06], answering a question of Thurston. The existence part was proved before in [Lab92]. Partial results for higher dimension manifolds with smooth boundary are in [Sch01b].

By contrast to the statements of the two theorems above, which concern topologically simple manifolds, it seems to be not possible to make a straightforward polyhedral analogue of the smooth statement of [Sch06],

because some convex polyhedral metrics could be realized by boundaries of 3-manifolds containing some “pleated” faces (instead of totally geodesic ones), see [Sch03, Sch04].

For each of the problems mentioned above, a “dual” problem can be considered as well. For example in the case of the boundary of the convex core, the dual statement implies bending laminations on surfaces.

Such considerations can be translated in terms of “Globally hyperbolic AdS manifolds” (a kind of analogue of quasi-Fuchsian manifolds modeled on anti-de Sitter geometry). In this setting, characterization of the dual metric on the boundary of the convex core is done in [BS10], and characterization of the induced metric is the subject of [Diaon]. For more details about this subject, we send the reader to the review paper [Sch03] and to [Mes07, ABB⁺07].

Recently, the kind of statements mentioned here are extended from the setting of manifolds to the one of “manifolds with particles” (cone-manifolds with singularities along certain geodesics, in particular time-like in the Lorentzian setting), see [BS09, LS09]. Such objects appear naturally in the variational proofs for realization theorems, and they are also related to circle patterns problems on surfaces with conical singularities.

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